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Homogeneous Solutions of the Heat Equation

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We complete a characterization of homogeneous solutions of the heat equation begun by D. V. Widder. We determine regions of convergence for expansions of temperature functions in terms of the homogeneous solutions.

1. INTRODUCTION

A solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \tag{1}$$

is called homogeneous of degree α if

$$u(\lambda x, \lambda^2 t)=\lambda^\alpha u(x, t) \tag{2}$$

for all $\lambda > 0$.

Two important sequences of homogeneous solutions of the heat equation were introduced by Rosenbloom and Widder in [4]. These are $\{v_n\}$ which are homogeneous of degree n and $\{w_n\}$ which are homogeneous of degree $-n-1$ ($n \geqslant 0$). We recall the definition of v_n and w_n below. Rosenbloom and Widder gave necessary and sufficient conditions for convergence of expansions of functions in terms of $\{v_n\}$ and $\{w_n\}$. The underlying idea is an analogy between these expansions and the expansions of analytic functions in Laurent series, i.e., in terms of $\{z^n\}$, $-\infty < n < \infty$. A summary of investigations into this analogy can be found in [6]. One observes however that the heat equation is the analog of Laplace's equation in this context, and for a given integer n Laplace's equation has two independent solutions with homogeneity n: $\mathrm{Re}(z^n)$ and $\mathrm{Im}(z^n)$. One is therefore led to

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investigate all homogeneous solutions of the heat equation with a given homogeneity. This was done—with the exception of two cases—by Widder in [5]. In this note we supply the two missing functions. It may be interesting to note that the key is the Hilbert transform, which is the classical tool connecting the real and imaginary parts of some analytic functions.

Consider the ordinary differential equation

$$2tu'' + xu' - \alpha u = 0, \tag{3}$$

where u' denotes the derivative of u with respect to x.

It is proved in [5] that any twice differentiable function which satisfies any two of (1), (2), or (3) also satisfies the third. It follows that a complete characterization of all solutions with homogeneity α consists in finding two independent solutions of (3) with homogeneity α. Moreover, once we have a solution of (3) homogeneous of any degree, we can obtain a solution of the correct homogeneity by multiplying it by an appropriate power of t.

There is, of course, a considerable amount of latitude in choosing two independent solutions of (3). The functions one gets are used as basis elements for representing solutions of the heat equation. The choices of particular solutions of (3) are influenced by the properties of the resulting bases. In particular, the theory of the Hilbert transform enables us to give necessary and sufficient conditions for convergence of expansions in these bases.

For $n = 0, 1, \ldots$ the "heat polynomials" of degree n are the unique polynomials of degree n which satisfy the heat equation, have coefficient of x^n equal to unity, and have homogeneity n. They are defined [4] as

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{t^k x^{n-2k}}{k!(n-2k)!}. \tag{4}$$

Part of the interest in the $v_n(x, t)$ stems from their relationship to the Hermite polynomials. For $n = 0, 1, \ldots$ define the nth Hermite polynomial orthogonal with respect to the measure $e^{-x^2/2}\,dx$ as

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. \tag{5}$$

Then [4]

$$v_n(x, t) = (-2t)^{n/2} H_n\left(\frac{x}{\sqrt{-2t}}\right). \tag{6}$$

A second independent solution of (3) can be determined by applying a reduction of order argument to (3) with $\alpha = n$ using $v_n(x, t)$ as the first solution. This gives for $t > 0$ and $x > 0$ [5]

$$h_n(x, t) = n!\,(2t)^n v_n(x, t) \int_x^\infty \frac{k(y, t)}{v_n^2(y, t)}\,dy, \tag{7}$$

where

$$k(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \tag{8}$$

is the Gauss–Weierstrass kernel. In [5] it is shown that $h_n(x, t)$ has the simpler representation

$$h_n(x, t) = \int_x^\infty k(y, t)(y - x)^n \, dy, \tag{9}$$

so that $(n!)^{-1} h_n(x, t)$ is the $(n+1)$st integral of $k(x, t)$. Note that the condition $t > 0$ is necessary for the convergence of both integrals. In the first integral, the reason is the existence of the roots of the $v_n(x, t)$. Since the Hermite polynomials have n real roots symmetric about the origin, $v_n(x, t)$ has both positive and negative roots if $t < 0$.

For negative integer homogeneities $\alpha = -n - 1 < 0$ one set of solutions of (3) are the "associated heat polynomials" given in [4],

$$w_n(x, t) = k(x, t)\, v_n\left(\frac{x}{t}, \frac{-1}{t}\right) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \left(\frac{2}{t}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2t}}\right). \tag{10}$$

Observe that $w_n(x, t)$ is the Appell transform of $v_n(x, t)$.

The sequences $\{w_n\}$ and $\{v_n\}$ are biorthogonal for $0 < t < \infty$:

$$\int_R v_m(x, -t)\, w_n(x, t)\, dx = \delta_{mn}. \tag{11}$$

For $t < 0$, a second set of solutions of (3) is obtained by applying a reduction of order argument to $w_n(x, t)$:

$$g_n(x, t) = \frac{-n!\, w_n(x, t)}{t^{n+1}} \int_x^\infty \frac{k(y, t)}{w_n^2(y, t)} \, dy. \tag{12}$$

The following simpler representation is given in [5]:

$$g_n(x, t) = 2 \int_0^\infty e^{-xy + ty^2} y^n \, dy. \tag{13}$$

The condition $t < 0$ is necessary for the convergence of the integrals in (12) and (13).

Another way of obtaining $g_n(x, t)$ is to take the Appell transform of $h_n(x, t)$.

We summarize Widder's linearly independent solutions of (3) which satisfy the homogeneity condition (2) with integer n in the following table:

	Homogeneous of degree n	Homogeneous of degree $-n-1$
$t > 0$	$v_n; h_n$	w_n
$t < 0$	v_n	$w_n; g_n$

A second independent solution is missing for $t<0$, positive homogeneity, and for $t>0$, negative homogeneity. As we have seen before, reduction of order fails to give a second set of homogeneous solutions in these cases. Using some recent results about the Hilbert transform of the Gaussian [1] we are able to give the missing solutions. We then prove necessary and sufficient conditions for the convergence of expansions in terms of the homogeneous solutions.

2. A Characterization of Homogeneous Solutions

The Hilbert transform of $f \in L^p(R)$, $1 \leqslant p < \infty$, is defined a.e. by

$$\mathscr{H}f(x) = p.v. \frac{1}{\pi} \int_R \frac{f(s)}{x-s}\, ds. \tag{14}$$

For $f(x, t) \in L^p(R, dx)$, let $\mathscr{H}f(x, t)$ denote the Hilbert transform of $f(x, t)$ with respect to the first variable.

Theorem 1. *If $u(x, t) \in L^p(R, dx)$, $1 \leqslant p < \infty$, is a solution of* (2) *and* (3) *in the strip $\sigma_1 < t < \sigma_2$, if also $xu'(x, t)$, $u''(x, t) \in L^p(R, dx)$, and $\lim_{|x| \to \infty} u(x, t) = 0$ for $\sigma_1 < t < \sigma_2$, then $\mathscr{H}u(x, t)$ is also a solution of* (2) *and* (3) *in the strip $\sigma_1 < t < \sigma_2$.*

Proof. Since the Hilbert transform commutes with dilations, (2) will be satisfied for $\mathscr{H}u(x, t)$. Thus, we only need to prove that $\mathscr{H}u(x, t)$ will satisfy (3). We have

$$\begin{aligned} &\mathscr{H}\{2tu''(x, t) + xu'(x, t) - \alpha u(x, t)\} \\ &\quad = 2t\mathscr{H}\{u''(x, t)\} + \mathscr{H}\{xu'(x, t)\} - \alpha \mathscr{H}u(x, t) \\ &\quad = 2t(\mathscr{H}u)''(x, t) + \mathscr{H}\{xu'(x, t)\} - \alpha \mathscr{H}u(x, t). \end{aligned}$$

Using Theorem 1 in [2] we have

$$\begin{aligned} \mathscr{H}\{xu'(x, t)\} &= x\mathscr{H}\{u'(x, t)\} + \mathscr{H}\{xu'(x, t)\}(0) \\ &= x(\mathscr{H}u)'(x, t) - p.v. \frac{1}{\pi} \int_R \frac{su'(s, t)}{s}\, ds \\ &= x(\mathscr{H}u)'(x, t) - \frac{1}{\pi} \int_R u'(s, t)\, ds \\ &= x(\mathscr{H}u)'(x, t). \end{aligned}$$

Therefore, we have

$$2t(\mathscr{H}u)''(x,t)+x(\mathscr{H}u)'(x,t)-\alpha\mathscr{H}u(x,t)=0.$$

This concludes the proof of the theorem.

Concerning the conditions of the theorem, observe that if $u(x)$, $u'(x)$ are absolutely continuous, if also $xu'(x)$, $u''(x)\in L^p(R)$, $1\leqslant p<\infty$, and $\lim_{|x|\to\infty}u(x)=0$, then an application of Hardy's inequality shows that $u(x)\in L^p(R)$.

The next result may be of independent interest.

Let $\mathscr{G}(x)$ denote the Gaussian $1/\sqrt{2\pi}\,e^{-x^2/2}$. In [1] it is proved that

$$\mathscr{H}\mathscr{G}(x)=\mathscr{S}(x)=\frac{1}{\pi}e^{-x^2/2}\int_0^x e^{u^2/2}\,du. \tag{15}$$

THEOREM 2. *Let $H_n(x)$ denote the Hermite polynomials with respect to the measure $\mathscr{G}(x)$. For $n=0,1,\ldots$ we have*

$$\mathscr{H}(H_n(x)\,\mathscr{G}(x))=H_n(x)\,\mathscr{S}(x)-P_{n-1}(x), \tag{16}$$

where $P_{-1}(x)=0$ and for $n=0,1,\ldots,$

$$P_n(x)=\frac{1}{\pi}\sum_{j=0}^{[n/2]}(-1)^j\frac{(n-j)!\,H_{n-2j}(x)}{(n-2j)!}. \tag{17}$$

Proof. For $n=0$ this is (15). For integer $n\geqslant 1$, $H_n'=nH_{n-1}$, so that

$$\begin{aligned}
&\mathscr{H}\{H_n(x)\,\mathscr{G}(x)\}\\
&=-\mathscr{H}\left\{(-1)^{n-1}\frac{d}{dx}\frac{d^{n-1}}{dx^{n-1}}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\}\\
&=-\frac{d}{dx}\mathscr{H}\{H_{n-1}(x)\,\mathscr{G}(x)\}\\
&=-\frac{d}{dx}[H_{n-1}(x)\,\mathscr{S}(x)-P_{n-2}(x)]\\
&=-[H_{n-1}'(x)\,\mathscr{S}(x)+H_{n-1}(x)\,\mathscr{S}'(x)-P_{n-2}'(x)]\\
&=-(n-1)\,H_{n-2}(x)\,\mathscr{S}(x)+xH_{n-1}(x)\,\mathscr{S}(x)-\frac{1}{\pi}H_{n-1}(x)+P_{n-2}'(x)\\
&=H_n(x)\,\mathscr{S}(x)-\frac{1}{\pi}H_{n-1}(x)+P_{n-2}'(x).
\end{aligned}$$

For $n=0, 1$, it can be directly verified that $(1/\pi) H_n(x) - P'_{n-1}(x) = P_n(x)$. For $n \geqslant 2$,

$$\begin{aligned} P'_{n-1}(x) &= \frac{1}{\pi} \sum_{j=0}^{[(n-1)/2]} (-1)^j \frac{(n-1-j)!\, H'_{n-1-2j}(x)}{(n-1-2j)!} \\ &= \frac{1}{\pi} \sum_{j=0}^{[(n-2)/2]} (-1)^j \frac{(n-1-j)!\, H_{n-2-2j}(x)}{(n-2-2j)!} \\ &= -\frac{1}{\pi} \sum_{j=1}^{[n/2]} (-1)^j \frac{(n-j)!\, H_{n-2j}(x)}{(n-2j)!}. \end{aligned}$$

Therefore

$$\frac{1}{\pi} H_n(x) - P'_{n-1}(x) = \frac{1}{\pi} \sum_{j=0}^{[n/2]} (-1)^j \frac{(n-j)!\, H_{n-2j}(x)}{(n-2j)!} = P_n(x), \tag{18}$$

and the proof is complete.

Recall for $t>0$ one set of solutions of (2) and (3) for homogeneities $-n-1<0$ are the $\{w_n\}$ which are in all $L^p(R, dx)$, $1 \leqslant p \leqslant \infty$. Theorems 1 and 2 enable us to find a second set of homogeneous solutions.

THEOREM 3. *For $t>0$, $n=0, 1, \ldots$, $\mathscr{H} w_n(x, t)$ satisfies the heat equation, has homogeneity $\alpha = -n-1$, and is linearly independent of $w_n(x, t)$. $\mathscr{H} w_n(x, t)$ has the representation*

$$\mathscr{H} w_n(x, t) = \frac{1}{\sqrt{2t}} \left(\frac{2}{t}\right)^{n/2} \left[H_n\left(\frac{x}{\sqrt{2t}}\right) \mathscr{S}\left(\frac{x}{\sqrt{2t}}\right) - P_{n-1}\left(\frac{x}{\sqrt{2t}}\right) \right], \tag{19}$$

where P_n is the polynomial of Theorem 2.

Proof. By Theorem 1, $\mathscr{H} w_n(x, t)$ satisfies the heat equation and has homogeneity $-n-1$. If $g \in L^p(R)$, $1 \leqslant p < \infty$, is not the boundary value of a function analytic in the upper half plane, then g and $\mathscr{H} g$ are linearly independent. Therefore w_n and $\mathscr{H} w_n$ are linearly independent.

By Theorem 2 and the dilation invariance of the Hilbert transform,

$$\begin{aligned} \mathscr{H} w_n(x, t) &= \frac{1}{\sqrt{4\pi t}} \left(\frac{2}{t}\right)^{n/2} \mathscr{H} \left\{ e^{-u^2/4t} H_n\left(\frac{u}{\sqrt{2t}}\right) \right\} (x) \\ &= \frac{1}{\sqrt{2t}} \left(\frac{2}{t}\right)^{n/2} \mathscr{H} \{ \mathscr{G}(u)\, H_n(u) \} \left(\frac{x}{\sqrt{2t}}\right) \\ &= \frac{1}{\sqrt{2t}} \left(\frac{2}{t}\right)^{n/2} \left[H_n\left(\frac{x}{\sqrt{2t}}\right) \mathscr{S}\left(\frac{x}{\sqrt{2t}}\right) - P_{n-1}\left(\frac{x}{\sqrt{2t}}\right) \right]. \end{aligned}$$

This concludes the proof.

We turn our attention to the case $t<0$, integer homogeneity $n\geqslant 0$. Recall the definition of the Appell transform:

$$\mathscr{A}u(x, t)=k(x, t)\, u\left(\frac{x}{t}, \frac{-1}{t}\right). \tag{20}$$

If $u(x, t)$ is a solution of the heat equation in a domain D, then $\mathscr{A}u(x, t)$ is a solution of the heat equation in the domain

$$\tilde{D}=\left\{(x, t)\colon \left(\frac{x}{t}, \frac{-1}{t}\right)\in D\right\}. \tag{21}$$

If $u(x, t)$ is homogeneous of degree α then $\mathscr{A}u(x, t)$ is homogeneous of degree $-\alpha-1$:

$$\begin{aligned}\mathscr{A}u(\lambda x, \lambda^2 t)&=k(\lambda x, \lambda^2 t)\, u\left(\frac{\lambda x}{\lambda^2 t}, \frac{-1}{\lambda^2 t}\right)\\ &=\lambda^{-1}k(x, t)\, \lambda^{-\alpha}u\left(\frac{x}{t}, \frac{-1}{t}\right)\\ &=\lambda^{-\alpha-1}\mathscr{A}u(x, t).\end{aligned}$$

Therefore

$$\mathscr{A}\mathscr{H}w_n(x, t)=k(x, t)\, \mathscr{H}w_n\left(\frac{x}{t}, \frac{-1}{t}\right)=t^{n+1}k(x, t)\, \mathscr{H}w_n(x, -t) \tag{22}$$

gives a second solution in $t<0$ for integer homogeneity $n\geqslant 0$. It is clearly linearly independent of the $v_n(x, t)$.

This completes the classification of all homogeneous solutions of the heat equation with integer valued homogeneity. We summarize:

solutions of the heat equation with integer homogeneity $n\geqslant 0$:

$t>0$

$$v_n(x, t)=n!\sum_{k=0}^{[n/2]}\frac{t^k x^{n-2k}}{k!\,(n-2k)!}=(-2t)^{n/2}H_n\left(\frac{x}{\sqrt{-2t}}\right)$$

$$h_n(x, t)=n!\,2^n t^n v_n(x, t)\int_x^\infty \frac{k(y, t)}{v_n^2(y, t)}\,dy=\int_x^\infty k(y, t)(y-x)^n\,dy$$

$t<0$

$$v_n(x, t)=n!\sum_{k=0}^{[n/2]}\frac{t^k x^{n-2k}}{k!\,(n-2k)!}=(-2t)^{n/2}H_n\left(\frac{x}{\sqrt{-2t}}\right)$$

$$\mathscr{A}\mathscr{H}w_n(x, t)=t^{n+1}k(x, t)\, \mathscr{H}w_n(x, -t)$$

solutions of the heat equation with integer homogeneity $-n-1<0$:

$t>0$

$$w_n(x,t)=k(x,t)\,v_n(x/t,-1/t)=k(x,t)\left(\frac{2}{t}\right)^{n/2}H_n\left(\frac{x}{\sqrt{2t}}\right)$$

$$\mathscr{H}w_n(x,t)=\frac{1}{\sqrt{2t}}\left(\frac{2}{t}\right)^{n/2}\left[H_n\left(\frac{x}{\sqrt{2t}}\right)\mathscr{S}\left(\frac{x}{\sqrt{2t}}\right)-P_{n-1}\left(\frac{x}{\sqrt{2t}}\right)\right]$$

$t<0$

$$w_n(x,t)=k(x,t)\,v_n(x/t,-1/t)=k(x,t)\left(\frac{2}{t}\right)^{n/2}H_n\left(\frac{x}{\sqrt{2t}}\right)$$

$$g_n(x,t)=\frac{-n!\,w_n(x,t)}{t^{n+1}}\int_x^\infty\frac{k(y,t)}{w_n^2(y,t)}\,dy=2\int_0^\infty e^{-xy+ty^2}y^n\,dy.$$

3. Regions of Convergence

We are interested in the L^p convergence of the expansions $\sum c_n\mathscr{H}w_n(x,t)$ and $\sum c_n\mathscr{A}\mathscr{H}w_n(x,t)$; see Theorems 7 and 8 below. The proofs in the cases $p=1$ and $p=\infty$ are complicated by the fact that $\mathscr{S}(x)$ is not an $L^1(R)$ function. We therefore need some preliminary results.

Define the Weyl half derivative as

$$D_t^{1/2}f(t)=\frac{i}{\sqrt{\pi}}\int_t^\infty f'(u)(u-t)^{-1/2}\,du. \tag{23}$$

We have from Lemma 2 in [3]:
For $x>0$ and $\beta>1$, if for all $t>0$

$$|f'(t)|\leqslant\min\left\{\frac{1}{x^\beta},\frac{1}{t^{\beta/2}}\right\}, \tag{24}$$

then

$$|D_t^{1/2}f(t)|\leqslant C(\beta)\cdot\min\left\{\frac{1}{x^{\beta-1}},\frac{1}{t^{(\beta-1)/2}}\right\}, \tag{25}$$

where

$$C(\beta)=\max\left\{\Gamma\left(\frac{\beta-1}{2}\right)\Big/\Gamma\left(\frac{\beta}{2}\right),\ \frac{1}{\sqrt{\pi}}\left(2+\frac{2}{\beta-1}\right)\right\}. \tag{26}$$

Lemma 4. *For $n=0,1,\ldots,t>0$, there exist constants B_n such that*

$$|\mathscr{H}w_n(x,t)|\leqslant A_n\cdot\min\left\{\frac{B_n}{|x|^{n+1}},\frac{1}{t^{(n+1)/2}}\right\}, \tag{27}$$

where

$$A_n = O\left(n^{3/4}\left(\frac{2n}{e}\right)^{n/2}\right). \tag{28}$$

Proof. The case $n=0$ follows from L'Hôpital's. For $n \geqslant 1$, it is shown in [4] that there exists an absolute constant A such that

$$|w_n(x, t)| \leqslant Ae^{-x^2/8t}\left(\frac{2n}{et}\right)^{n/2} \frac{n^{1/4}}{t^{1/2}}. \tag{29}$$

Since for any $\beta > 0$

$$\frac{1}{t^\beta} e^{-x^2/8t} \leqslant \left(\frac{8\beta}{e}\right)^\beta \frac{1}{|x|^{2\beta}} \tag{30}$$

we have from (29)

$$|w_n(x, t)| \leqslant An^{1/4}\left(\frac{2n}{e}\right)^{n/2} \cdot \min\left\{\frac{B_n}{|x|^{n+1}}, \frac{1}{t^{(n+1)/2}}\right\}. \tag{31}$$

Since $D_x w_n(x, t) = -\frac{1}{2} w_{n+1}(x, t)$ (see [4]), we have

$$\begin{aligned} |D_t w_n(x, t)| &= |D_x^2 w_n(x, t)| \\ &= \tfrac{1}{4}|w_{n+2}(x, t)| \\ &\leqslant A(n+2)^{1/4}\left(\frac{2(n+2)}{e}\right)^{(n+2)/2} \cdot \min\left\{\frac{B_n}{|x|^{n+3}}, \frac{1}{t^{(n+3)/2}}\right\}. \end{aligned}$$

For $0 < t_0 < t$ (see [4]), we have

$$w_n(x, t) = \int_R k(x-y, t-t_0)\, w_n(y, t_0)\, dy. \tag{32}$$

Therefore, by Theorem 6 in [3] we have

$$D_x \mathscr{H} w_n(x, t) = iD_t^{1/2} w_n(x, t) \tag{33}$$

so that

$$\begin{aligned} |\mathscr{H} w_{n+1}(x, t)| &= \tfrac{1}{2}|D_x \mathscr{H} w_n(x, t)| \\ &= \tfrac{1}{2}|D_t^{1/2} w_n(x, t)| \\ &\leqslant A_{n+1} \cdot \min\left\{\frac{B_{n+1}}{|x|^{n+2}}, \frac{1}{t^{(n+2)/2}}\right\}. \end{aligned}$$

COROLLARY 5. *For* $t>0$, $n=0, 1, \ldots$

$$|\mathscr{A}\mathscr{H}w_n(x, -t)| \leqslant A_n t^{n+1/2} e^{x^2/4t} \cdot \min\left\{\frac{B_n}{|x|^{n+1}}, \frac{1}{t^{(n+1)/2}}\right\}, \tag{34}$$

where A_n *and* B_n *are the constants of Lemma* 4.

Consider temperature functions in the span of the $\{w_n\}$:

$$u(x, t) = \sum_{n=0}^{\infty} c_n w_n(x, t). \tag{35}$$

Rosenbloom and Widder, in [4], showed that if $\sum_{n=0}^{\infty} a_n v_n(x, -\sigma)$ converges at all points $x \in E$, where E is a set of positive measure, then $|a_n| = O((e/2n\sigma)^{n/2})$. (The claim in [4] is for $E=[a, b]$ but their proof works equally well in the more general case.) Since $w_n(x, \sigma) = \mathscr{A} v_n(x, \sigma) = k(x, \sigma)\, v_n(x/\sigma, -1/\sigma)$ it follows that if (35) converges for $t=\sigma>0$ and all $x \in E$ then

$$|c_n| = O\left(\left(\frac{\sigma e}{2n}\right)^{n/2}\right). \tag{36}$$

Conversely, if (36) holds, then (35) converges absolutely and uniformly in half planes $t \geqslant t_0 > \sigma$ (see [4]).

THEOREM 6. *If* (36) *holds, then* (35) *converges in* $L^p(R, dx)$ *norm,* $1 \leqslant p \leqslant \infty$, *uniformly in half planes* $t \geqslant t_0 > \sigma$.

Conversely, if for some $t_0 > 0$ (35) *converges in some* $L^p(R, dx)$ *norm,* $1 \leqslant p \leqslant \infty$, *then* (36) *holds for all* $\sigma > t_0$.

Proof. For $n \geqslant 1$ and $t \geqslant t_0 > \sigma$, if c_n satisfies (36), we have from (29) for $1 \leqslant p \leqslant \infty$

$$\sum_{n=1}^{\infty} |c_n| \cdot \|w_n(\cdot, t)\|_p \leqslant A \|e^{-x^2/8t}\|_p \sum_{n=1}^{\infty} |c_n| \frac{n^{1/4}}{t^{1/2}} \left(\frac{2n}{et}\right)^{n/2}$$
$$\leqslant A_{p, t_0} \sum_{n=1}^{\infty} n^{1/4} \left(\frac{\sigma}{t_0}\right)^{n/2} < \infty,$$

so that we have uniform $L^p(R, dx)$ convergence for $t \geqslant t_0$.

For the converse it suffices to show that convergence in $L^p(R, dx)$ norm for some $t_0 > 0$ implies pointwise convergence for $t > t_0$. Let $S_N(x, t) = \sum_{n=0}^{N} c_n w_n(x, t)$. By (32) we have

$$|S_N(x, t) - S_M(x, t)| \leqslant \|k(\cdot, t-t_0)\|_{p'} \|S_N(\cdot, t_0) - S_M(\cdot, t_0)\|_p,$$

where $1/p + 1/p' = 1$. The last term converges to zero as $N, M \to \infty$, and the theorem is proved.

Note that since $D_x^{(k)} w_n(x, t) = (-\frac{1}{2})^k w_{n+k}(x, t)$, $n, k = 0, 1, 2, \ldots$ ([4]), we have L^p convergence of the expansion of $D_x^{(k)} u(x, t)$ obtained by termwise differentiation of (35).

THEOREM 7. *Suppose for some* $\sigma > 0$:

$$|c_n| = O\left(\left(\frac{\sigma e}{2n}\right)^{n/2}\right). \tag{37}$$

For $t > \sigma$, *let*

$$u(x, t) = \sum_{n=1}^{\infty} c_n w_n(x, t). \tag{38}$$

Then

$$\mathscr{H} u(x, t) = \sum_{n=1}^{\infty} c_n \mathscr{H} w_n(x, t) \tag{39}$$

converges in $L^p(R, dx)$ *norm*, $1 \leqslant p \leqslant \infty$, *uniformly in half planes* $t \geqslant t_0 > \sigma$.

Conversely, if for some $t_0 > 0$ (39) *converges in some* $L^p(R, dx)$ *norm*, $1 \leqslant p \leqslant \infty$, *then* (37) *holds for all* $\sigma > t_0$.

Proof. For $1 < p < \infty$ the L^p convergence of (39) follows trivially from the L^p continuity of the Hilbert transform. The point of the proof below is therefore the cases $p = 1$ and $p = \infty$.

If (37) holds, then

$$\sum_{n=1}^{\infty} c_n w_{n-1}(x, t)$$

converges in $L^p(R, dx)$, $1 \leqslant p \leqslant \infty$, for each $t > \sigma$. By (32), if $n \geqslant 1$ and $0 < t_0 < t$, since $D_x w_n(x, t) = (-1/2)\, w_{n+1}(x, t)$, we have

$$\begin{aligned} \mathscr{H} w_n(x, t) &= \int_R \mathscr{S}(x-y, t-t_0)\, w_n(y, t_0)\, dy \\ &= -2 \int_R \mathscr{S}'(x-y, t-t_0)\, w_{n-1}(y, t_0)\, dy, \end{aligned} \tag{40}$$

where $\mathscr{S}(x, t) = \mathscr{H} k(x, t) = \mathscr{H} w_0(x, t)$, and differentiation is with respect to the first variable. By Lemma 4, $\mathscr{S}'(x, t) = -\frac{1}{2} \mathscr{H} w_1(x, t) \in L^p(R, dx)$, $1 \leqslant p \leqslant \infty$.

Let $\mathscr{H} S_N(x, t) = \sum_{n=0}^{N} c_n \mathscr{H} w_n(x, t)$. Then

$$\|\mathscr{H} S_N(\cdot, t) - \mathscr{H} S_M(\cdot, t)\|_p \leqslant \|\mathscr{S}'(\cdot, t-t_0)\|_1 \cdot \left\| \sum_{n=M+1}^{N} c_n w_{n-1}(\cdot, t_0) \right\|_p.$$

The convergence is uniform for $t \geqslant t_1 > t_0 > \sigma$.

Conversely, if for some $t_0 > 0$ (39) converges in some $L^p(R, dx)$ we have from (32)

$$w_n(x, t) = -\int_R \mathscr{S}(x-y, t-t_0)\, \mathscr{H}w_n(y, t_0)\, dy$$
$$= 2\int_R \mathscr{S}'(x-y, t-t_0)\, \mathscr{H}w_{n-1}(y, t_0)\, dy \qquad (41)$$

so that for $1 \leqslant p \leqslant \infty$, $t > t_0$,

$$\left| \sum_{n=M+1}^{N} c_{n-1} w_n(x, t) \right| \leqslant 2\, \|\mathscr{S}'(\cdot, t-t_0)\|_{p'} \cdot \left\| \sum_{n=M+1}^{N} c_{n-1} \mathscr{H}w_{n-1}(\cdot, t_0) \right\|_p .$$

Therefore, by Theorem 6,

$$|c_{n-1}| = O\left(\left(\frac{\sigma e}{2n} \right)^{n/2} \right) \qquad (42)$$

for all $\sigma > t_0$. Therefore (37) holds for all $\sigma > t_0$.

Note, since $D_x^{(k)} \mathscr{H}w_n(x, t) = (-\frac{1}{2})^k \mathscr{H}w_{n+k}(x, t)$, we have L^p convergence of the expansion of $D_x^{(k)} \mathscr{H}u(x, t)$ obtained by termwise differentiation of (39).

Since for $t > 0$, $\mathscr{A}\mathscr{H}w_n(x, -t) = k(x, -t)\, \mathscr{H}w_n(-x/t, 1/t)$ we determine the region of convergence of the series $\sum c_n \mathscr{A}\mathscr{H}w_n(x, t)$ using Theorem 7. We get:

THEOREM 8. *If for some* $\sigma > 0$

$$|c_n| = O\left(\left(\frac{e}{2n\sigma} \right)^{n/2} \right) \qquad (43)$$

then the series

$$u(x, t) = \sum_{n=1}^{\infty} c_n \mathscr{A}\mathscr{H}w_n(x, -t) \qquad (44)$$

converges uniformly in strips $0 \leqslant t \leqslant t_0 < \sigma$. *Furthermore, the series for* $u(x, t)/k(x, -t)$ *converges uniformly in* $L^p(R, dx)$, $1 \leqslant p \leqslant \infty$, *in strips* $0 \leqslant t \leqslant t_0 < \sigma$.

Conversely, if for some $t_0 > 0$ *the series for* $u(x, t)/k(x, -t)$ *converges in some* $L^p(R, dx)$, $1 \leqslant p \leqslant \infty$, *then* (43) *holds for all* $\sigma < t_0$.

REFERENCES

1. A. P. CALDERÓN AND Y. SAGHER, The Hilbert Transform of the Gaussian, to appear.
2. E. KOCHNEFF, Y. SAGHER, AND R. TAN, On a theorem of Akhiezer, to appear.
3. E. KOCHNEFF AND Y. SAGHER, Conjugate temperatures, to appear.
4. P. C. ROSENBLOOM AND D. V. WIDDER, Expansions in terms of heat polynomials and associated functions, *Trans. Am. Math. Soc.* **92** (1959), 220–266.
5. D. V. WIDDER, Expansions in terms of the homogeneous solutions of the heat equation, *in* "Orthogonal Expansions and Their Continuous Analogues," Southern Illinois Univ. Press, 1967.
6. D. V. WIDDER, "The Heat Equation," Academic Press, San Diego, 1975.